The Tachyon Field below the mass barrier *

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Abstract

We consider a tachyon field whose Fourier components correspond to spatial momenta with modulus smaller than the mass parameter. The plane wave solutions have them a time evolution which is a real exponential. The field is quantized and the solution of the eigenvalue problem for the Hamiltonian leads to the evaluation of the vacuum expectation value of products of field operators. The propagator turns out to be half-advanced and half-retarded. This completes the proof [4] that the total propagator is the Wheeler Green function [4,7].
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1 Introduction

For a normal particle (bradyon), the Klein-Gordon equation

$$(\Box + m^2)\varphi = 0 \tag{1}$$

leads to mass-energy relation of the form

$$E = (\vec{k}^2 + m^2)^{\frac{1}{2}} \tag{2}$$

which implies that E is real for any real value of $\vec{k}$.

Instead, for the tachyon field we have

$$(\Box - \mu^2)\phi = 0 \tag{3}$$

so that the elementary solution is

$$\phi = e^{i(wt - \vec{k}\cdot\vec{x})} \quad \text{with} \quad w = (\tilde{k}^2 - \mu^2)^{\frac{1}{2}} \tag{4}$$

and w is real only when $\vec{k}^2 \geq \mu^2$. For $\vec{k}$ below the mass barrier ($\vec{k}^2 < \mu^2$), w is pure imaginary and the exponential in [4] blows up either for $t \to -\infty$ or for $t \to +\infty$.In general this region of $\vec{k}$ is discarded (See Ref. [1] and [2]) and only values such that $\vec{k}^2 \geq \mu^2$ is accepted.

However the situation here is similar to the case of the ordinary quantum-mechanical description of the movement of a free particle that find in its way a potential barrier higher than its kinetic energy. The real solutions inside the potential cannot be ignored. In our case the "mass" squared μ^2 plays the role of the potential barrier.

It is nowadays that tachyons cannot appear asymptotically as free particles without impairing unitarity [3], but this does not mean that they have to be barred from consideration. They should be forbidden to propagate freely but they can be allowed to exist as transient virtual modes.

This point of view was adopted in a previous paper for $\vec{k}^2 \geq \mu^2$ [4]. Now we are going to examine some consequences of considering (3) and (4) for $\vec{k}^2 < \mu^2$.

It is perhaps convenient at this point to warm about a possible misinterpretation of (4). It should be clear that an imaginary value of w means that this quantity cannot be the energy of any state of the field. As matter of fact we will see that, even if the exponential in (4) is a solution of the field equation (3), there is no actual eigenstate with particle-like properties which can be considered similar to those of bradyons.

In this paper only values of $\vec{k}$ such that $\vec{k}^2 < \mu^2$ will be considered.

2 Quantization

Any real solution of (3) can be written as

$$\phi = \phi_1 + \phi_2 \tag{5}$$

$$\phi_1 = \frac{1}{(2\pi)^{\frac{N-1}{2}}} \int \frac{dk}{\sqrt{w}} \left(b_k^1 e^{-wt} + c_k^1 e^{wt} \right) \cos \vec{k} \cdot \vec{r} \tag{6}$$

$$\phi_1 = \frac{1}{(2\pi)^{\frac{N-1}{2}}} \int \frac{dk}{\sqrt{w}} \left(b_k^2 e^{-wt} + c_k^2 e^{wt} \right) \sin \vec{k} \cdot \vec{r} \tag{7}$$

where from now on we define

$$w = |(\vec{k}^2 - \mu^2)^{\frac{1}{2}}| = (\mu^2 - \vec{k}^2)^{\frac{1}{2}} \tag{8}$$

and all integrations are to be carried out over the region $\vec{k}^2 < \mu^2$.

The Lagrangian for the field is:

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi \partial^\mu \phi + \frac{1}{2}\mu^2 \phi^2 \tag{9}$$

The total Hamiltonian can be written

$$\mathcal{H} = \mathcal{H}_1 + \mathcal{H}_2 \tag{10}$$

$$\mathcal{H}_j = \int dk \frac{w}{2} (b_k^j c_k^j + c_k^j b_k^j) \tag{11}$$

It is now easy to see that canonical quantization of the field leads to the commutation relations

$$[b_k^l, c_{k'}^m] = i \delta^{lm} \delta(\vec{k} - \vec{k}') \tag{12}$$

The eigenvalue problem for $\mathcal{H}_j$ has been discussed in Ref. [5]. For the sake of completeness we shall give some results that will be needed afterwards.

For each degree of freedom (each $\vec{k}$), the Hamiltonian has the form

$$h = \frac{1}{2}(qp + pq) = qp - \frac{i}{2} \qquad [q, p] = i \tag{13}$$

The eigenvalue equation is

$$h\psi = \left(qp - \frac{i}{2} \right) \psi = E\psi \tag{14}$$

or, in the usual coordinate representation

$$-iq\frac{d}{dq}\psi = \left(E + \frac{i}{2}\right)\psi \tag{15}$$

whose solution we write as

$$\psi^{E} = \frac{1}{\sqrt{2\pi}} q_{+}^{iE-\frac{1}{2}} \qquad -\infty < E < +\infty \tag{16}$$

[$q_{+}^{\alpha} = q^{\alpha}\Theta(q)$ and similar solutions with $q_{-}^{\alpha} = |q|^{\alpha}\Theta(-q)$] The spectrum is continuous and runs the real energy axis from $-\infty$ to $+\infty$. The normalization is such that:

$$(\psi^{E}, \psi^{E'}) = \frac{1}{2\pi}\int_{0}^{\infty} dq q^{-iE-\frac{1}{2}} q^{iE'-\frac{1}{2}} = \frac{1}{2\pi}\int_{0}^{\infty} \frac{dq}{q} q^{i(E'-E)} =$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} dy\; e^{i(E'-E)y} = \delta(E - E') \tag{17}$$

($y = \ln q$) The solutions (16) are then similar to the plane wave solutions for free particles. They have been used in quantum optics to describe the time solutions "squeezed states" and also to evaluate the Shannon entropy for the degenerate parametric amplifier whose Hamiltonian can be cast into the form (11) (See Ref. [13]).

The orthogonality relations are contained (for $E \neq E'$) in the general expression (see Ref. [6])

$$\int_{0}^{\infty} dq\; q^{\alpha} = 0 \qquad (\alpha \neq -1) \tag{18}$$

These formulae show that the mean values of q^{μ} (and p^{μ}) are 0 for $\mu \neq 0$

$$(\psi^{E}, q^{\mu}\psi^{E'}) = \frac{1}{2\pi}\int_{0}^{\infty} dq q^{-iE-\frac{1}{2}} q^{\mu} q^{iE-\frac{1}{2}} = \frac{1}{2\pi}\int_{0}^{\infty} dq\; q^{\mu-1} = 0 \quad (\mu \neq 0) \tag{19}$$

On the other hand we have:

$$(\psi^{E}, pq\psi^{E'}) = \frac{1}{2\pi}\int_{0}^{\infty} dq q^{-iE-\frac{1}{2}} (-i)\frac{d}{dq} q q^{iE'-\frac{1}{2}} =$$

$$\left(E' - \frac{i}{2}\right)\delta(E - E') = \left(E - \frac{i}{2}\right)(\psi^E, \psi^{E'}) \tag{20}$$

and of course

$$(\psi^E, qp\psi^{E'}) = \left(E + \frac{i}{2}\right)(\psi^E, \psi^{E'}) \tag{21}$$

We shall now write the zero energy mean values using the notation $\psi^{E=0} \equiv |0>$. Then (19)-(21) reads

$$< 0|q\mu|0 >= 0 =< 0|p\mu|0 > \qquad (\mu \neq 0) \tag{22}$$

$$< 0|pq\mu|0 >= -\frac{i}{2} < 0|0 >= - < 0|qp\mu|0 > \tag{23}$$

3 Vacuum Expectation Values of Products of Field Operators

Taking into account (6), (12), (22) y (23), we can evaluate the mean value for products of the ϕ_1 field at different points (we shall take $< 0|0 >= 1$)

$$< 0|\phi_1(x)\phi_1(y)|0 >=$$

$$\frac{1}{(2\pi)^{N-1}} \int \frac{dk}{\sqrt{w}} \int \frac{dk'}{\sqrt{w'}} < 0|b^1_k c^1_k|0 > \cos\vec{k}\cdot\vec{x}\cos\vec{k}\cdot\vec{y} e^{-(wx_0 - w' y_0)} +$$

$$< 0|c^1_k b^1_k|0 > \cos\vec{k}\cdot\vec{x}\cos\vec{k}\cdot\vec{y} e^{(w' x_0 - w y_0)} =$$

$$< 0|\phi_1(x)\phi_1(y)|0 >= \frac{i}{(2\pi)^{N-1}} \int \frac{dk}{2w} \cos\vec{k}\cdot\vec{x}\cos\vec{k}\cdot\vec{y}(e^{-w(x_0-y_0)} - e^{w(x_0-y_0)}) \tag{24}$$

Analogously

$$< 0|\phi_2(x)\phi_2(y)|0 >= \frac{i}{(2\pi)^{N-1}} \int \frac{dk}{2w} \sin\vec{k}\cdot\vec{x}\sin\vec{k}\cdot\vec{y}(e^{-w(x_0-y_0)} - e^{w(x_0-y_0)}) \tag{25}$$

Adding (23) to (24) we get

$$< 0|\phi(x)\phi(y)|0 >= \frac{i}{(2\pi)^{N-1}} \int \frac{dk}{2w} e^{-i\vec{k}\cdot(\vec{x}-\vec{y})}(e^{-w(x_0-y_0)} - e^{w(x_0-y_0)}) \tag{26}$$

from which we obtain

$$< 0|T\phi(x)\phi(y)|0 >=$$

$$\frac{i}{(2\pi)^{N-1}}\int\frac{dk}{2w}e^{-i\vec{k}\cdot(\vec{x}-\vec{y})}(e^{-w(x_0-y_0)}-e^{w(x_0-y_0)})Sg(x_0-y_0)$$

$$\frac{i}{(2\pi)^{N-1}}\int\frac{dk}{2w}e^{-i\vec{k}\cdot(\vec{x}-\vec{y})}\sinh|x_0-y_0| = \frac{1}{2\pi}W(x-y) \tag{27}$$

The Green function (27) is the Wheeler propagator defined in Ref. [7]. Further if we note that (Ref. [8], 3.354, p. 312)

$$\int\limits_{-\infty}^{\infty}\frac{e^{-ik_0t}}{k_0^2+w^2} = \frac{\pi}{w}e^{-wt}$$

then we see that $W(x)$ is the "Fourier transform" of $(k_0^2+w^2)^{-1} = (k_0^2-\vec{k}^2+\mu^2)^{-1}$ where the k_0 integration path can be considered to be the superposition of two branches. In one of them both poles are left below the path integration and in the other branch are left above the path. In other words, $W(x)$ is half-retarded plus half-advanced (see also Ref. [9] and [10])

$$W(x) = \frac{1}{2}\Delta_R(x)+\frac{1}{2}\Delta_A(x) \tag{28}$$

4 Lorentz Invariance

The generators of the Poincaré group can be constructed of the energy-momentum tensor:

$$\mathcal{T}_\mu^\nu = -\partial_\mu\partial_\nu\phi+\frac{1}{2}(\partial_\rho\partial_\rho\phi-\mu^2\phi^2)\delta_\mu^\nu \tag{29}$$

With (5)-(7) we find that

$$\mathcal{P}_0 = \frac{i}{2}\int dk\ k_0[\{b_k^1, c_k^1\}+\{b_k^2, c_k^2\}] \qquad k_0 = iw$$

$$\mathcal{P}_i = \frac{1}{2}\int dk\ k_i[\{b_k^2, c_k^1\}-\{b_k^1, c_k^2\}] \qquad 1\leq i\leq 3 \tag{30}$$

From the angular momentum tensor

$$\mathcal{M}_{\mu\nu}^\rho = x_\mu\mathcal{T}_\nu^\rho - x_\nu\mathcal{T}_\mu^\rho \tag{31}$$

we obtain, in particular, the Lorentz boost

$$\mathcal{M}_{0i} = \frac{1}{2}\int dk\, w[\{\partial_i b_k^2, c_k^1\} - \{\partial_i b_k^1, c_k^2\}] \qquad \partial_i \equiv \frac{\partial}{\partial k_i} \tag{32}$$

It is a matter of tedious algebra to verify that the operators $\mathcal{P}_\mu$ and $\mathcal{M}_{\mu\nu}$ obey the commutation relations of teh Poincaré algebra.

To verify that ϕ transforms as a scalar under the transformations of the Lorentz group generated by $\mathcal{M}_{\mu\nu}$, it is necessary to take into account that the Fourier transform of ϕ is an analytic generalized function (analytic distribution; see Ref. [11]), so that the path of integration [in (6) and (7)] can be deformed in the complex $\vec{k}$ plane. But this proof is actually not needed, as the field of the tachyon disappears from all matrix elements (in contradistinction to the case of bradyons) For this reason the tachyonic field should be considered more an auxiliary concept than a real entity. The only trace of the field ϕ leaves is the Wheeler propagator, which, as explained in Sec. 3 is the Fourier transform of $(k_0^2 - \vec{k}^2 + \mu^2)^{-1} = (k^2 + \mu^2)^{-1}$

The complete propagator is Lorentz invariant, as shown by K. Kamoi and S. Kamefuchi in Ref. [12] for the retarded and advanced Green function (cf. eq. (28)).

5 Discussion

When tachyons are forbidden to propagate asymptotically as free particles, all values of $\vec{k}$ are acceptable in principle and $\vec{k}$ space divides naturally into two regions: (a) the sphere $\vec{k}^2 < \mu^2$ and (b) the complement $\vec{k}^2 \geq \mu^2$. The case (b) was considered in Ref. [4]. There it was shown that the coefficients of elementary solutions are increasing and decreasing operators. None of them annihilates the vacuum, which is an eigenstate of the Hamiltonian with zero energy and zero momentum When $\vec{k}^2 > \mu^2$ and $\vec{k}^2 \to \mu^2$ the energy momentum spectrum is discrete but becomes more and more dense. For $\vec{k}^2 < \mu^2$ we find a continuum. For (a) or (b), the propagator is found to be half-advanced plus half-retarded, the only difference being that in (b) both poles in the k_0 plane are real while in (a) they are pure imaginary.

The tachyon field is related to complex mass fields through the Wheeler propagator, which appears in a natural way (see also Ref. [7]).

Furthermore, this is practically, the only Green function which is compatible with the suppression of the asymptotic free modes of the field.

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